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# New Perspectives on Kac Ring Models

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It is proposed that the type of model first suggested by Kac in connection with problems of nonequilibrium statistical mechanics can be generalized and modified so that it can be directly applied to cellular automata. It is further noted that these same models can be used to illuminate some basic questions in the interpretation of quantum mechanics.

**KEY WORDS:**

## 1. PROLOGUE

In his remarkable autobiographical notes, Mark Kac quotes a statement by Uhlenbeck that "In Physics one must follow a Master."[1] As Kac later wrote, it was Uhlenbeck who introduced him to the problems of statistical physics.[2] Indeed, Kac's major contributions to statistical physics were direct outgrowths of suggestions first made by Uhlenbeck. Kac's brilliant solution of the Ehrenfest model was in direct response to Uhlenbeck's urging; his major contributions to the Ising model followed brief but inspired tutoring by Uhlenbeck on that topic.

There is a third area, less well known, certainly less intensely elaborated, where Uhlenbeck's urging again inspired Kac to interesting and potentially important ideas. Uhlenbeck, as a student of Ehrenfest, was deeply interested in the question of the approach to thermal equilibrium of statistical systems. The role and fundamental significance of the Boltzmann

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The content of this paper shows more clearly than any words how much physics owes to the insight, vigor, and brilliance of Mark Kac. It cannot show the personal debt this author owes to him. Throughout his life he showed compassion, kindness, and consideration. Few days go by that I do not think of him, his work, his humanity.

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equation—its status within physics—were of paramount concern throughout his life. In Uhlenbeck's investigations, he (although not very explicitly) introduced the category of Master equations as distinguished from Liouville equations. In the Master equation approach the dynamics is described stochastically, for example, by prescribing fundamental transition probabilities. In the Louville description, the dynamics is described precisely, totally deterministically, while the needed macroscopic quantities are obtained from subsequent statistical averages.

Kac became interested in the precise relation between the Master and Liouville equations. (Uhlenbeck tended to stress the solutions of the Master-type equations such as the Boltzmann equation and the Kramers equation.) In order to investigate that relation, Kac invented a very ingenious model, the Kac ring model,[3] in which both the exact dynamical treatment with a subsequent averaging and the statistical treatment could be carried out rigorously, making a direct comparison between exact solutions possible. Kac wrote only one paper on this model.[3] He was very pleased that his example showed that the two approaches (Master and Liouville approach) were not completely equivalent, but he never returned the the model or to the general connection.

Later, extensions and elaborations of this model were studied by many people.[4–6] Apart from pointing out a number of interesting and unexpected features of the Kac model, they conformed that in general the two statistical treatments, the Liouville and Master treatments, were not equivalent. Although perhaps not generally known in this degree of detail or in this particular context, the overall drift of these results did not come as a great surprise. These special results verified that for these contrived models, with a "poverty-stricken" dynamics, a typical Master's level (Boltmann level) treatment did not contain all the dynamical information. Their relevance for realistic systems, or for the foundations of statistical mechanics, seemed rather minor. Apart from emphasizing that in general the approach to a unique equilibrium is not inevitable and certainly not universal, the study of models had little impact.

The practitioners of statistical mechanics were well aware when and where a Master equation description that would lead to the usual equilibrium state was legitimate and effective. It is the purpose of this note to suggest that the Kac models, or modifications inspired by these models, might be helpful and illuminating in several other areas. There is currently a great deal of interest in the temporal evolution of cellular automata. Section 4 will indicate briefly just how the techniques developed from the Kac ring models are well adapted for an analytic (as contrasted to a computational) treatment of cellular automata. Another distinct area in which Kac models can play an important role is in the construction of what

might be called "minimal quantum theories." It is well known that in spite of the phenomenal success of quantum theory, there are a number of physicists and plilosophers who seek alternate, pseudoclassical causal theories that would nevertheless reproduce the main features of quantum theory. There exists a quantum version of the Kac ring model,[5] which contains just the additivity of probability amplitudes. Section 5 contains some results and conjectures about he possibility (or impossibility!) of constructing classical hidden variables for such systems.

These are the only "new perspectives of Kac models" touched upon in this paper. But it is worth pointing out that the precise relation between continuous and discrete descriptions of lattice gauge theories and the connection between continuous and discrete chaos all have substantial formal and conceptual overlap with the simple Kac ring models. Thus, Mark Kac's simple example "constructed in part to irritate those who think non equilibrium statistical mechanics is finished"[8] has a surprising number of unexpected potentialities for interesting physics.

## 2. BACKGROUND. THE FORMAL STRUCTURE OF KAC MODELS

a. The original Kac model and its variations are described in detail in Refs. 4 and 5. Here just the notation and some basic equations will be recalled. The model schematically describes impurity scattering where the scattering objects (particles or balls) are capable of just two states. The underlying geometry is a one-dimensional chain whose lattice points are indicated by $j = 1,..., n$. During each time interval, all particles placed on all the lattice points move along the chain. The scattering centers are at fixed locations; a complete characterization is provided by a set of $\varepsilon_j$ variables $j = 1,..., n$, where $\varepsilon = +1$ if there is no scattering center at $j$ and $\varepsilon_j = -1$ if there *is* a scattering center at $j$.

In the simplest Kac models, the scattered objects are capable of just two states (spin up and spin down, or white and blue). Each particle is described by a variable $\eta_j(t) = +1$ or $-1$, which gives the state of the object at location $j$ at time $t$. The system at time $t$ is described by the set $\{\eta_j(t)\}$. Because of the motion of the objects and the action of the scattering centers, the set $\{\eta_j(t)\}$ changes in the course of time. Just what that change is depends on the nature of the *assumed* scattering dynamics. In the original Kac model it was assumed that when an object leaves a scattering center (it has been scattered) its state is required to change. This leads to the equation of motion

$$\eta_{j+1}(t+1) = \varepsilon_j \eta_j(t) \tag{1}$$

Since the impurity distribution is fixed, the $\varepsilon_j$ variables are independent of the time and unaffected by the dynamics.

This level of description is the *Newtonian* level; Eq. (1) can be trivially solved; all results will of course depend on the set $\{\varepsilon\}$. Statistical ideas enter, because typically and in practice the set $\{\varepsilon\}$ is not given as such. All that is usually known is average or probability information. In almost all Kac models the probability distribution of the scattering centers is given by

$$\text{Prob}(\varepsilon) = \tfrac{1}{2} + \tfrac{1}{2}(1 - 2\mu)\varepsilon \tag{2}$$

This is an uncorrelated probability distribution

$$P(\varepsilon_j \varepsilon_k) = (\text{Prob}\ \varepsilon_j)(\text{Prob}\ \varepsilon_k) \tag{3}$$

The Liouville level description is obtained by studying the temporal evolution of an observable $Q$; this will depend explicitly on $\varepsilon_1, \ldots, \varepsilon_n$. The macroscopic or observed value is given by

$$Q_{\text{obs}}(t) = \sum_{\varepsilon_1} \cdots \sum_{\varepsilon_n} \text{Prob}(\varepsilon_1, \ldots, \varepsilon_n)\, Q(\varepsilon_1, \ldots, \varepsilon_n, t) \tag{4}$$

In general the solution of the Liouville equation is needed to obtain $Q(\varepsilon_1, \ldots, \varepsilon_n, t)$.

The equations for the one- and two-particle (Liouville) distribution functions are

$$f_j^{(1)}(\alpha, t) = f_{j-1}^{(1)}(\varepsilon_{j-1}\alpha, t-1) \tag{5a}$$

$$f_{jk}^{(2)}(\alpha, \beta, t) = f_{j-1,k-1}^{(2)}(\varepsilon_{j-1}\alpha, \varepsilon_{k-1}\beta, t-1) \tag{5b}$$

$f_j^{(1)}(\alpha, t)$ is the probability that the object at site $j$ at time $t$ is in the state $\alpha$. As anticipated, $f^{(1)}$ and $f^{(2)}$ do depend on $\{\varepsilon\}$. The pattern outlined here is the characteristic *exact*, rigorous procedure.

The Master or Boltzmann description (adapted to the specific impurity distribution) starts from probability distributions $F_j^{(1)}(\alpha, t)$, $F_{jk}^{(2)}(\alpha, \beta, t)$, which incorporate (2) and (3) from the start:

$$F_j^{(1)}(\alpha, t) = (1-\mu)\, F_{j-1}^{(1)}(\alpha, t-1) + \mu[1 - F_{j-1}^{(1)}(\alpha, t-1)] \tag{6}$$

$\mu$ functions via (2) as the average density of the scatterers. No further averaging is needed. The main results of the classical analysis are that:

I. In general $F_j^{(1)}(\alpha, t) = \langle f_j^{(1)}(\alpha, t) \rangle_\varepsilon$.

II. In general $F_{jk}^{(2)}(\alpha, \beta, t) \neq \langle f_{jk}^{(2)}(\alpha, \beta, t) \rangle_\varepsilon$.

III. The limit $t \to \infty$ of $f_{jk}^{(2)}(\alpha, \beta, t)$ is not always 1/4; the limit $t \to \infty$ of $F_{jk}^{(2)}$ is 1/4, as an elementary analysis would yield.

$\langle f \rangle_\varepsilon$ is the ensemble average of $f$

b. An extension of the original Kac model retains the same basic ideas, but introduces a classical probability $\lambda$ that when scattered an object will change its state with probability $\lambda$. The same two methods of treatment can again be carried out. The equation for $f^{(1)}$ becomes instead of (5)

$$f_j(\alpha, t) = -\lambda\varepsilon_{j-1} f_{j-1}(\alpha, t-1) + \tfrac{1}{2}(1-\varepsilon_{j-1})\lambda + f_{j-1}(\alpha_1 t - 1) \tag{7}$$

As should be anticipated, the equation again depends explicitly on $\varepsilon$. The Boltzman type function $F_j^{(1)}$ satisfies instead of (6)

$$\begin{aligned} F_j(\alpha, t) = {} & F_{j-1}(\alpha, t-1)(1-\mu) + F_{j-1}(\alpha, t-1)\,\mu(1-\alpha) \\ & + [1 - F_{j-1}(\alpha, t-1)]\,\mu\lambda \end{aligned} \tag{7a}$$

The analysis now shows (5) that

I. $F_j^{(1)}(\alpha, t) = \langle f_j^{(1)}(\alpha, t)\rangle_\varepsilon$.

II. $F_{jk}^{(2)}(\alpha, \beta, t) \neq \langle f_{jk}^{(2)}(a, \beta, t)\rangle_\varepsilon$.

III. $\lim_{t\to\infty} F_{jk}^{(2)} = \lim_{t\to\infty} \langle f_{jk}^{(2)}\rangle_\varepsilon = 1/4$.

This type of model is of a somewhat mixed, hybrid character. It contains stochastic elements explicitly through the probability aspects of the scattering mechanism (the parameter $\alpha$), but it also contains deterministic dynamical aspects, through the $\varepsilon$ variables. The distinct statistical treatments, the Liouville and Master methods, refer to the different treatments of the $\varepsilon$ variables. In the Liouville method the $\varepsilon$ dependence of the dynamics is retained throughout, treated exactly, with only a final averaging of the observables. In the Master method the dynamics itself is based on the average or probability character of the $\varepsilon$ variables. The important observation is that the results are *not* the same.

c. It was shown in Ref. 5 that it is possible to construct a "quantum" extension of these models. The crucial feature is to describe the state of the objects by a two-component probability amplitude $\langle\eta_j(t)|$. The time evolution of the system [the analogue of Eq. (1)] is given by the action of an operator $E$, which acts in the two-dimensional color subspace:

$$|\eta_{j+1}(t+1)\rangle = E_j\,|\eta_j(t)\rangle \tag{8}$$

The choice of $E$ is determined by the model rules. If the rules are chosen to be the same as those described in the classical stochastic model in Section 2b, $E_j$ is given by

$$E_j = \tfrac{1}{2}(1+\varepsilon_j)1 + \tfrac{1}{2}(1-\varepsilon_j)S_j \tag{9a}$$

$$S_j = \begin{pmatrix} (1-\lambda)^{1/2} & \lambda^{1/2} \\ \lambda^{1/2} & (1-\lambda)^{1/2} \end{pmatrix} \tag{9b}$$

This, with the usual quantum interpretation, quarantees that after a scattering event, the color (state) will change with probability $\lambda$ as required. This quantum model can be analyzed using the Master equation or the Liouville equation methods. Instead of dealing with the classical distribution functions $f$ and $F$ of the previous section, one has now to employ the reduced density matrices. The analysis (see especially Ref. 5) leads to the following results.

I. The quantum Master description is in all respects (one-, two-, and many-particle distributions) identical to the classical master description of Section 2b.

II. The one-particle quantum density matrix (Liouville description) is different for all times from the classical Liouville description. The density matrix does approach equilibrium, but in an oscillatory manner, with a single fixed frequency $\nu_0$ which depends on $\mu$ and $\lambda$.

III. For higher correlation functions, there is an oscillatory behavior containing several fixed frequencies. There is a limiting state, but it is generally not the equilibrium state. This behavior is in total contrast to the classical Liouville behavior.

IV. There are persistent correlations, which do not depend on the time. Again, classically there are no such terms in the Liouville description of the stochastic model.

V. There are memory effects. Apart from the one-particle density matrix, all higher correlation functions retain a memory of the initial configuration.

Thus, the three levels of description of the stochastic model—classical Master, classical Liouville, quantum Liouville—actually lead to qualitative differences in the phenomena predicted.

## 3. DISCUSSION AND EXTENSIONS

a. It is not particularly difficult to dream up a number of generalizations of these models. For example, it would be possible to investigate models where the $\varepsilon$ variables depend on the time explicitly. The simplest situation would be one where $\varepsilon_p$ changes from $+1$ to $-1$ in a given manner, for example,

$$\begin{aligned} \varepsilon_p(t) &= 1 \qquad 2nT_p < (2n+1)T_p \\ \varepsilon_p(t) &= -1 \qquad (2n+1)T_p < (2n+2)T_p \end{aligned} \tag{10}$$

This corresponds to a location-dependent periodicity of the presence (absence) of the scattering centers. Althoug suggested by the Kac model, this is quite a different physical system from the original model. The natural questions to ask are not the relation between Master and Liouville equations, but rather the existence of solutions periodic in time. This type of problem is in fact not unrelated to the process of wave transmission through a random medium. The solution of the fundamental equation is just

$$\eta_p(t) = \varepsilon_{p-1}(t-1)\,\varepsilon_{p-2}(t-2)\cdots\varepsilon_{p-t}\eta_{p-t}(0) \tag{11}$$

with $\varepsilon_p(t)$ given by (10). This expression can be analyzed; one could compute

$$\Gamma(t) = \frac{1}{n}\sum_p \eta(t) \tag{12}$$

and study the existence of the limit $t \to \infty$ of $\Gamma(t)$; but this type of model does not yield any striking new insights in statistical mechanics, although it might be useful for modeling nerve transmission phenomena.

b. It is of some interest (especially in connection with the question of whether lattice gauge theories approach thermal equilibrium) to investigate this particular modification of the Kac models. The model is (to start out with) classical, but the dynamical variables at each point are complex numbers of modulus one

$$\eta_p(t) = e^{i\theta p(t)} \tag{13}$$

The effect of a scattering is to change the phase of the dynamical variable according to

$$\eta_{p+1}(t+1) = e^{i(1-\varepsilon p)\alpha/2}\eta_p(t) \tag{14}$$

This reduces to the original Kac model of $\alpha = \pi$.

This system does allow an $\varepsilon$ averaging; from (14) it follows that

$$\eta_p(t) = e^{it\alpha/2}\, e^{-i\varepsilon_{p-1}\alpha/2}\, e^{-i\varepsilon_{p-2}\alpha/2}\cdots e^{-i\varepsilon_{p-t}\alpha/2}\eta_{p-t}(0) \tag{15}$$

In the original Kac model, the quantity $\Gamma$ defined by (12) gives the difference between the number of white balls and black balls, or the net polarization of the beam. In this case with $\eta_p$ complex, the interpretation is less direct, but one can still ask for limit as $t \to \infty$ of $\langle \eta_p(t) \rangle_\varepsilon$. In any case $\Gamma \to 0$ implies equilibrium.

A little calculation shows

$$\langle \eta_p(t) \rangle_\varepsilon = (1 - \mu + \mu e^{\alpha})^t \eta_{p-t}(0) \tag{16}$$

If all objects start out in the same state,

$$\langle \Gamma(t) \rangle_{\varepsilon} = (1 - \mu + \mu e^{\alpha})^{t} \tag{17}$$

It is not difficult to show that the absolute value of $1 - \mu + \mu e^{i\alpha}$ is less than 1 for all $\alpha$. This in conjunction with (17) shows that this system always goes to equilibrium. A many-dimensional generalization where the time evolution is determined by elements of $O(n)$ rather than $O(1)$ would be interesting to consider. (It can be analyzed in detail.)

c. One of the serious disadvantages of the Kac models is their linear character. This means that many important statistical feasures have no counterpart in these models. Thus, there is nothing like a coupled hierarchy of equations; there are no mixing properties. This makes it difficult to take all the qualitative features exhibited by these models terribly seriously. Attempts to introduce more realistic features tend to complicate the models to such an extent that exact solutions are no longer possible. If, for example, one retains the one-dimensional models, but allows the scattering centers to recoil so that the model possesses genuine collisions, no exact solutions are known.

It is possible to construct a example of interacting Kac rings (or interacting Kac ladders). An example of such a system would be given by two isomorphic Kac ladders, each provided with its own set of scattering centers, described, respectively, by the sets $\{\varepsilon_p\}$ and $\{\varepsilon_p^1\}$ (Fig. 1). The rules are the same as before, except that if a point $p$ in I corresponding to a point $p^1$ has a scattering center, the subsequent color in II switches to that of I. It is easy to show that the equations of motion are genuinely coupled:

$$\begin{aligned} \eta_{p+1}(t+1) &= \tfrac{1}{2}(1+\varepsilon_p^1)\,\varepsilon_p \eta_p(t) + \tfrac{1}{2}(1-\varepsilon_p^1)\,\eta_p^1 \\ \eta_{p+1}^1(t+1) &= \tfrac{1}{2}(1+\varepsilon_p)\,\varepsilon_p^1 \eta_p^1(t) + \tfrac{1}{2}(1-\varepsilon_p)\,\eta_p \end{aligned} \tag{18}$$

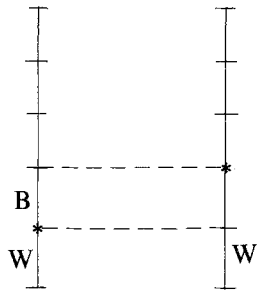


Fig. 1

This gives an interaction, but a very trivial one. One shows directly that $S_p(t) = \eta_p(t) + \eta_p^1(t)$ satisfies

$$S_{p+1}(t+1) = \tfrac{1}{2} S_p(t)(1 + \varepsilon_p \varepsilon_p^1) \tag{19}$$

or

$$S(t) = (\tfrac{1}{2})^t (1 + \varepsilon_{p-1}\varepsilon_{p-1}^1)(1 + \varepsilon_{p-2}\varepsilon_{p-2}^1) \cdots (1 + \varepsilon_{p-t}\varepsilon_{p-t}^1)\, S_{p-t}(0) \tag{20}$$

(20) shows that if somewhere along the chains the corresponding $\varepsilon$'s are such that $1 + \varepsilon_q \varepsilon_q^1 = 0$, from that point on $S$ remains zero, so that from then on $\eta_p + \eta_p^1 = 0$ and remains zero; all that can happen is an exchange of colors (or spins). This shows that this interacting system can evolve into a periodic behavior. The possibility for this to happen is the probability that $1 + \varepsilon_q \varepsilon_q^1 = 0$ for some $q$. If the marker density in I is $\mu$ and in II is $\mu^1$, then this probability is given by $\mu + \mu^1 - 2\mu\mu^1$. So the interaction has some effect, although it is certainly not dramatic.

## 4. KAC MODELS AND CELLULAR AUTOMATA

Phrased in a somewhat more general and abstract manner, the Kac models belong to a class of dynamical systems where both the time and the spatial variables are discrete. Underlying the model is a fixed and given geometric structure: a lattice. At the lattice points, dynamical variables are defined, which can only assume a finite member of specified (usually real) numerical values. Finally, a deterministic rule is supplied that really defines the dynamics of the system and gives the temporal progression of the system. In the Kac model, there is one additional, final operation, which is the averaging over the interaction sites or the scattering centers.

It will be evident that the structure so defined has more than a superficial similarity to a cellular automaton. These objects, as defined by Wolfram,[9] possess the following properties:

1. They are *discrete in time.*
2. They are *discrete in space*, consisting of a discrete grid of spatial cells.
3. They assume *discrete* states; each cell has a finite number of values.
4. They are *homogeneous*; all cells are equivalent and are arranged in a regular pattern.
5. They allow *synchronous updating*; all cells are updated in synchrony, each depending on the previous values of neighboring cells.

6. The follow a *deterministic rule*; each cell value is updated according to a fixed deterministic rule.
7. They follow a *spatially local rule*; the rule at each site depends only on the values of a local neighborhood of sites around it.
8. The follow a *temporally local rule*; the rule for a new value of a site depends only on values for a fixed number of preceding steps (usually just one).

A sample formal example of a one-dimensional automaton is

$$\eta_i(t+1) = f(\eta_{i-1}(t), \eta_i(t), \eta_{i+1}(t1)) \tag{21}$$

where $f$ is a given function. It is evident that the Kac model, before the $\varepsilon$ averaging is carried out, is an example of a very trivial one-dimensional cellular automaton; it is not a homogeneous automaton, because of the presence of the scattering centers. It can be written in a form similar to (21), as

$$\eta_i(t+1) = f_i(\eta_i(t)) \tag{22}$$

This is in fact simpler than (21), but the dynamical rule in the Kac model is location-dependent. The $\varepsilon$ averaging process is not typically contained in the Wolfram categorization of cellular automata. Thus, the Liouville level of the Kac model is an inhomogeneous *one*-dimensional cellular automaton, to which is adjoined an averaging process.

By contrast, the Master description of the Kac model is a direct example of a probabilistic cellular automaton. This merely means that Wolfram's rule 6 is replaced by a probability statement; in turn, the description of the automaton itself is now in terms of distribution functions, or probabilities, in precise analogy to the Master description.

In spite of their simple formal structure, cellular automata can exhibit extremely involved, unexpected, and complex behavior, as is already noticeable in the Kac models. In fact, cellular automata are probably the most effective way to investigate complex behavior. One could—and in this respect the study of cellular automata goes much beyond statistical mechanics (although both investigate temporal or iterative behavior)—study questions of growth and extinction as in life games, or the establishment of geometrical patterns. Traditional mathematical methods have not done too well in these studies. Digital computers are the main tools, and are indispensable for the analysis of these systems. But it is perhaps not altogether pointless to try to merge the ideas and methods developed for the Kac models with the numerical methods used in cellular automata. The Master equation methods, i.e., the probablistic cellular automata, while smoothing out many dynamical details, give some

generally correct qualitative information—and they can be used in an analytic setting. One of the topics that might be studied in this manner is the lattice dependence of the evolutionary process. It is known from the experiences with cellular automata that there is an intricate and highly nontrivial relation between the possible lattice structures, the rules governing the temporal evolution, the function $f$, and the resulting dynamical behavior. It appears not too far-fetched that this relationship can be investigated using the techniques developed in the Kac models.

Another more ambitious program would be to investigate the phenomena associated with the temporal evolution in a changing geometry. In terms of the Kac models, this would involve in first instance a coupling of the location of the scattering centers and the dynamical variables of the type

$$\eta_i(t+1) = f_i(\eta_{\bar{i}}(t), \varepsilon_i(t)) \tag{23a}$$

$$\varepsilon_i(t+1) = g_i(\eta_{\bar{i}}(t), \varepsilon_i(t)) \tag{23b}$$

Here $\bar{i}$ stands for the set of neighbors of location $i$; $i$ could be in any dimension. In (22), the distribution of the scattering centers, which are the physically active sites, is dynamically coupled to the dynamical site variables. One might conjecture that the Master level description of this problem would resemble the behavior of an interacting system (friendly or unfriendly random walkers!).

A still more ambitious program has been recently initiated by Ilachinski[10] in which the dynamics and the underlying geometry themselves are coupled in a dynamical fashion. The topological structure of the lattice is now coupled to the local site value information. The analysis of this problem has so far been carried out using a computer, with rather unusual structures emerging. The problem suggested by (22) is not that ambitious, since the lattice sites $i$ are prescribed. However, the set $\{\varepsilon_i(t)\}$, where $\varepsilon_i = -1$, i.e., the set where the interactions actually take place, might well have a complicated and nontrivial structure. This might profitably be attacked using the ideas of this paper. The general moral of this section is that the detailed interrelation between Kac-type methodology and cellular automata is well worth exploring. Perhaps these analytical methods could be a useful adjunct to the obviously essential numerical techniques.

## 5. KAC MODELS AND MINIMAL QUANTUM THEORIES

It is interesting that the quantum Kac model described so extensively in Ref. 5 and mentioned briefly here in Section 2 contains the superposition principle of quantum theory and nothing else. For the rest its dynamics, in

mathematical structure, is precisely identical with the classical stochastic model also discussed in Ref. 5 and Section 2. Of course the quantum treatment deals with amplitues, while the classical discussion employs probability distribution functions. The additivity of the probability amplitudes is the single quantum element in the whole discussion. It is in this sense that this is a "minimum quantum model." Since the results of this quantum model are so striking and so different from the classical system, it is a natural place to test the possibility of a classical hidden variable model—which would reproduce (at least to some substantial precision) the qualitative quantum feature of this model.

A very obvious way to implement this suggestion would be to leave the whole model structure of the classical stochastic model intact, including the $\varepsilon$ variables, but assuming that the $\eta$ of variables, instead of just two values, could assume three values, say $\alpha_{-1}$, $\alpha_0$, $\alpha_+$. One could set up the classical formalism in this manner, and at some judicious juncture, declare just two of the variables to be classically observable and integrate or average over the remaining variable. There will be another parameter, which in principle could be adjusted as well. In the two-level classical model, the objects, upon scattering, change state with probability $\lambda$. This means that the one-particle distribution can be expressed in terms of a classical scattering matrix $S_d$ (which is distinct from the quantum scattering matrix). In fact the classical equation (7) can be written in terms of

$$S_d = \begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix} \tag{24}$$

In the three-level case, this becomes a $3 \times 3$ matrix, which upon the imposition of some natural symmetries assumes the form

$$S_d = \begin{pmatrix} 1-a-b & a & b \\ a & 1-2a & b \\ a & a & 1-a, b \end{pmatrix} \tag{25}$$

So the parameters of this classical model are $a$, $b$, and $\mu$ (the impurity density). The model to which this leads has been analyzed in detail, with the following results.

I. There are no persistent correlations.

II. Both the one- and two-particle distribution functions approach their equilibrium values of 1/3 and 1/9.

III. There is no memory of the initial state

IV. The Liouville and Master equation descriptions are equivalent on the one-particle level, but differ for the correlation functions. Generally, the solution in approaching these equilibrium values shows an exponential behavior (typically a sum of five exponentials). At values of $\mu$ larger than a critical density, the decaying solutions are modulated by a single field frequency.

As might be anticipated, there is no averaging process, nor an assignment of numerical values to $a$ and $b$ in (25), that will reproduce the quantum behavior. Obviously this does not prove that no such model is possible, but this classical three-level model does not do it.

(Although not directly relevant to the present discussion, one can also construct a three-level quantum model that shows all the complexities of the two-level quantum model. It shows in addition oscillatory behavior with a continuous range of frequencies; the frequency range is sensitively dependent on the density $\mu$ of the impurities.)

It is possible to construct a classical model that simulates more of the quantum features. In this model there are again three possible values for the $\eta$ variables; however, there are two types of impurities, or two distinct types of scattering centers. The mechanical equation remain

$$\eta_{p+1}(t+1) = \varepsilon_p \eta_p(t) \tag{26}$$

with

$$\begin{aligned} \eta_p &= \alpha_+ = e^{2\pi i/3} && \text{red} \\ \eta_p &= \alpha_0 = 1 && \text{white} \\ \eta_p &= \alpha_- = e^{4\pi i/3} && \text{blue} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \varepsilon_p &= \varepsilon_+ = e^{2\pi i/3} && \text{impurity of type 1} \\ &= \varepsilon_0 = 1 && \text{no impurity} \\ \varepsilon_- &= e^{4\pi i/3} && \text{impurity of type 2} \end{aligned} \tag{28}$$

This model has a complicated structure; it depends on the parameters $a$ and $b$ defined in (25) and the two types of densities $\mu$ (of $\varepsilon_+$) and $\nu$ of $(\varepsilon_-)$.

Both the stochastic and nonstochastic versions show oscillatious with a continuous range of frequencies; in fact, the same range as the quantum problem of the one-impurity-type model. However, in the stochastic two-impurity model there are no persistent correlations, no memory effects; both are present in the quantum model.

The new stochastic two-impurity model actually shows some similarities to the one-impurity quantum model. There are persistent correlations, memory effects, although they are qualitatively different.

It is not altogether excluded that a very complex scheme could be devised that has the same content as the minimal scheme suggested by the Kac model, but the evidence points in the opposite direction. The rather artificial, even Byzantine features that appear when one introduces more states, more objects, or more types of impurities in the hope that integration over them, or averaging these new elements, will replace quantum effects seem to guarantee that these systems will be contrived and complex. The physical properties of these systems, though they formally might be classical, are so artificial that they do little to provide a physical or pictorial intuition so necessary for an effective description of physical system.

It is remarkable that the Kac models constructed for altogether different purposes might yet be the most suitable systems to show the formal and physical superiority of quantum theory. We would conjecture that no classical model exists that can reproduce the physical results of the quantum Kac model. We guess that on the appropriate level of abstraction (not by an exhaustive analysis of special models) the proof would be simple.

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